DETERMINATION OF FORMAL CONFIDENCE INTERVALS OF THE REGRESSION LINES IN CASE OF LINEAR REGRESSION WITH BREAKPOINT (BP)

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Used in the SegReg program (software) for segmented regression at:
https://www.waterlog.info/segreg.htm

On website https://www.waterlog.info public domain, latest upload 20-11-2017

The two regression equations are:

RLa = Aa (X -AvXa) + AvYa

RLb = Ab (X -AvXb) + AvYb

where Aa is the regression coefficient to the left of BP, Ab is the regression coefficient to the right of BP, X is a distance along the horizontal axis, AvXa is the average of the X values smaller than BP, AvXb is the average of the X values larger than BP, AvYa is the average of the Y values of the data with X < BP, and AvYb is the average of the Y values of the data with X > BP.

Using ts = value of the variable in Student's distribution *) for the number of data employed at the desired confidence level, the **upper** confidence line to the **left** of BP is found from the relation:

$$(X\ ,\ Y1a + ts\ .\ StDevYc) \qquad (1)$$

where:

$$Y1a = Aa\ (X - AvXa) + AvYa \qquad (2)$$

$$StDevYc = \sqrt{\ \{ s_\gamma^2 + (X - AvXt)^2\ .\ StDevA^2 \}} \qquad (3)$$

with:

$$s_\gamma^2 = \{StDevYra^2\ (Na-1) + StDevYrb^2\ (Nb-1)\}\ \ /\ Nt\ (Nt-4) \qquad (4)$$

and:

$$StDevA^2 = \{StDevYra^2\ (Na-1) + StDevYrb^2\ (Nb-1)\}\ \ /\ (Nt-4)RedSumX^2 \qquad (5)$$

with:

$$RedSumX^2 = (Nt\text{-}1)(StDevX)^2 \qquad (6)$$

where: AvXt is the average of all X-data, StDevYra is the standard deviation of the residuals of Y values after regression (or of the distances between the Y values and Rla, StDevYr) to the left of BP, StDevYrb is the standard deviation of the residuals of Y values after regression (or of the distances between the Y values and Rlb, StDevYr) to the right of BP, Na is the number of data sets with X<BP, Nb is the number of data sets with X>BP, Nt is the total number of data sets (Nt = Na +Nb), and StDevX is the standard deviation of all the X-data (i.e. in all data sets)

Similarly, the **lower** confidence line to the **left** of BP is found from the relation:

$$(X\ ,\ Y1a - ts\ .\ StDevYc)$$

The **upper** confidence line to the **right** of BP is found from the relation:

(X, Y1b + ts . StDevYc)

where:

Y1b = Ab (X – AvXb) + AvYb

Similarly, the **lower** confidence line to the **right** of BP is found from the relation:

(X, Y1b – ts . StDevYc)

*) https://www.waterlog.info/t-tester.htm

EXAMPLE

From the example output file Dat.out (see below) we find:
Function type: 3
BP=3.06, Aa=As=0 (for data with X<BP), Ab=Ag= - 11 (for data with X>BP),
AvXa=1.5 (for data with X<BP), AvXb=6.75 (for data with X>BP),
AvXt =4.13 (for all data),
AvYa=140 (for data with X<BP), AvYb = 99.4 (for data with X>BP),
StDevYra=19.8 16.4 (for data with X<BP), StDevYrb=12.0 11.4 (for data with X>BP),
Na=12 (for data with X<BP), Nb=12 (for data with X>BP), Nt=24 (for all data),
StDevX=3.04 (for all data)

Note: for "all data" see the regression without BP

With these data of Dat.out it can be calculated that:

Eq (6): RedSumX^2 = (24-1) * 3.04^2 = 212.6
Eq (5): StDevA^2 = {19.8^2 *(12-1)+12.0^2 *(12-1)}/(24-4)*212.6={4312+1584}/4252=1.39
Eq (4): s_Y^2 = {19.8^2 *(12-1) + 12.0^2 *(12-1)} / 24*(24-4) = {4312+1584}/480 = 12.3
Eq (3): StDevYc = √ { 12.3 + $(X-4.13)^2$ * 1 .39}

Taking for example X=2 then StDevYc = √{9.14 + $(2 - 4.13)^2$ * 1.39} =
√{12.3 + 4.54 * 1.39} = √{9.14+6.31} = √18.61 = 4.31

Eq (2): Y1a = 0* (X-1.5) + 140 = 140
Function (1): for X=2 the upper confidence limit is 140 + ts * 4.31

Using the T-tester that can be downloaded from www.waterlog.info/t-tester.htm we find for degrees of freedom = 24 and Probability Pc (%) = 95 that ts = t-test value T = 1.71

Hence the **upper** 90% confidence limit of Y where X=2 is 140 + 1.71 * 4.31 = 140 + 7.37 = 147.4

The **lower** 90% confidence limit of Y where X=2 is 140 - 1.71 * 4.31 = 140 – 7.37 = 132.6

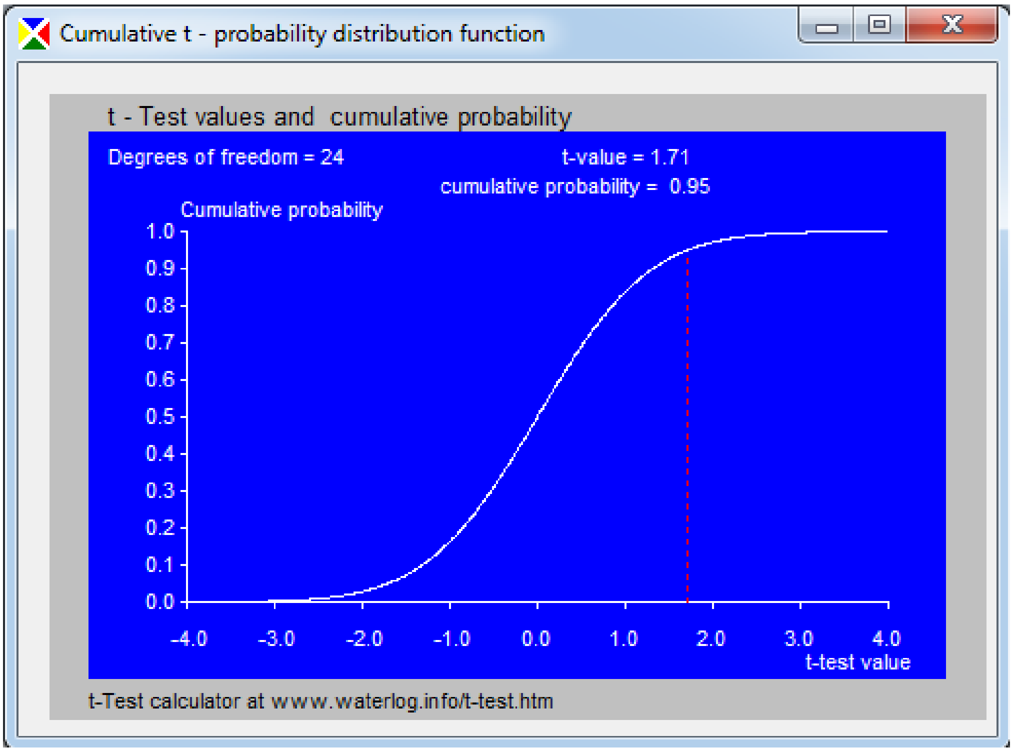
Note 1: Taking X = BP = 3.06 one finds from Eq (3) the value St.Dev.Ybp = 3.72, being the standard deviation of Y at BP, as can be seen in the example output file Dat.out

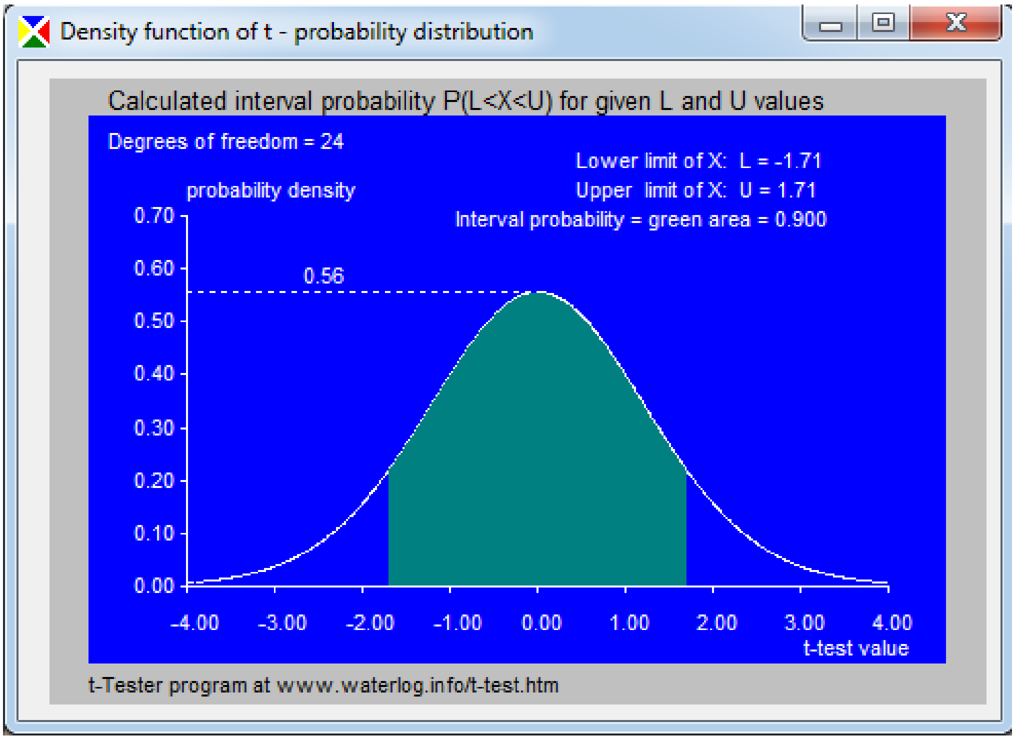
Note 2: The standard error of the breakpoint (St.Err.BP) is found from St.Dev.Ybp as follows

- Type 3 segmented regression: St.Err.BP =abs(St.Dev.Ybp / Ab)
  In this example St.Err.BP = 3.72 / 11 = 0.338
- Type 4 segmented regression: St.Err.BP = abs(St.Dev.Ybp / Aa)
- Type 2 segm. regression:
  St.Err.BP = 0.5*{abs(St.Dev.Ybp / Ab) + abs(St.Dev.Ybp / Aa)}

Using Student's ts value one can make a confidence interval for BP using St.Err.BP

Note 3: In the example, the upper and lower confidence limits span a 90% confidence interval ,with 5% probablity of exceedance (or 95% cumulative probability Pc) and 5% probability of non-exceedance (see figures below).




Density function of t - probability distribution
Calculated interval probability P(L<X<U) for given L and U values
Degrees of freedom = 24
Lower limit of X: L = -1.71
Upper limit of X: U = 1.71
Interval probability = green area = 0.900
probability density
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.56
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
t-test value
t-Tester program at www.waterlog.info/t-test.htm

EXAMPLE OUTPUT FILE (Dat.out)

```
Results of program SegReg for segmented linear regression of Y upon X.
Y is the dependent variable.
There can be different types of functions ranging from type 0 to type 6
 and (for types 2, 3 and 4) two methods of calculation.
 The types and methods are determined with the procedure of best fit.
 For explanations, use the symbols function in the output scroll menu.

 Name of this output file:  C:\SegReg\Dat.out
 Name of input file used :  C:\SegReg\Dat.inp

  No first title given
  No second title given
 Minimum confidence % : 90

Regression of Y upon X without breakpoint (BPx=Xmin).
X is the independent variable.
The table below gives the following series of values respectively:
    Breakpoint(BPz)    number of data       Av.Y          Av.X
    Regr.Coeff.(RC)    Corr.Coeff.Sq.      St.Dev.RC      Y(X=0)
      St.Dev.Y           St.Dev.Yr         St.Dev.X

 BPx=   0.00             2.40E+001        1.20E+002     4.13E+000
   -6.78E+000            5.36E-001        1.34E+000     1.48E+002
    2.81E+001            1.92E+001        3.04E+000

 Results of regression of Y upon X with optimal breakpoint (BPx)
 The second (Z) of two independent variables is used.
 The table below gives the following series of values respectively:
    Breakpoint(BPx)    number of data       Av.Y          Av.X
    Regr.Coeff.(RC)    Corr.Coeff.Sq.      St.Dev.RC       Y(X=0)
      St.Dev.Y           St.Dev.Yr         St.Dev.X
 for the data with X-values smaller and greater than BPx followed
 by the function parameters.

 Data with X < BPx :
 BPx=   3.06             1.20E+001        1.40E+002     1.50E+000
    9.95E+000            3.11E-001        4.69E+000     1.25E+002
    1.98E+001            1.64E+001        1.11E+000
 Data with X > BPx :
 BPx=   3.06             1.20E+001        9.94E+001     6.75E+000
   -8.79E+000            6.48E-001        2.05E+000     1.59E+002
    1.91E+001            1.14E+001        1.75E+000

 Parameters for function type 3 and method  2
   Slope > BPx             Ybp             N>/Nt       increase Yi
   -1.10E+001          1.40E+002        5.00E-001     2.03E+001
  St.Err.Slope>BPx     St.Err.BPx      St.Err.N>/Nt   St.Err.Yi
    2.17E+000          3.38E-001        1.02E-001     5.74E+000
   St.Dev.Yr > BPx    St.Dev.Yr < BPx   St.Dev.Ybp
    1.20E+001          1.98E+001        3.72E+000
   Slope < BPx        St.Err.Slope<BPx  Expl.Coeff.
    0.00E+000          5.64E+000        6.77E-001
```

SUMMARY OF THE Y-X REGRESSION.
Function type : 3 - first a horizontal segment, then sloping.
Calc. method : 2
See help functions on Intro tabsheet

Optimal breakpoint of X (BPx) : 3.060E+000

There are two regression equations:

when X is smaller than BPx: Y = AsX + Cs
when X is greater than BPx: Y = AgX + Cg

As = 0.00E+000 Ag = -1.10E+001
Cs = 1.40E+002 Cg = 1.74E+002

Serial	Yobs	X	Ycalc
1	1.14E+002	0.00E+000	1.40E+002
2	1.34E+002	0.00E+000	1.40E+002
3	1.36E+002	0.00E+000	1.40E+002
4	1.65E+002	1.50E+000	1.40E+002
5	1.57E+002	1.50E+000	1.40E+002
6	1.26E+002	1.50E+000	1.40E+002
7	1.38E+002	1.50E+000	1.40E+002
8	1.22E+002	1.50E+000	1.40E+002
9	1.16E+002	1.50E+000	1.40E+002
10	1.77E+002	3.00E+000	1.40E+002
11	1.42E+002	3.00E+000	1.40E+002
12	1.54E+002	3.00E+000	1.40E+002
13	1.32E+002	4.50E+000	1.24E+002
14	1.12E+002	4.50E+000	1.24E+002
15	1.22E+002	4.50E+000	1.24E+002
16	8.64E+001	6.00E+000	1.08E+002
17	1.23E+002	6.00E+000	1.08E+002
18	9.43E+001	6.00E+000	1.08E+002
19	8.86E+001	7.50E+000	9.11E+001
20	8.44E+001	7.50E+000	9.11E+001
21	1.09E+002	7.50E+000	9.11E+001
22	8.55E+001	9.00E+000	7.46E+001
23	7.77E+001	9.00E+000	7.46E+001
24	7.83E+001	9.00E+000	7.46E+001