

## Statistical significance of segmented linear regression with break-point.

### using variance analysis and F-tests.

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#### **Introduction**

In SegReg the significance of the break-point (BP) is indicated by the 90% confidence area around the BP as shown in the graphs. When the area remains within the data range, the break-point is significant, i.e. the BP gives a significant additional explanation compared to straightforward linear regression without a BP. Or, one can say that the BP analysis gives an improvement of the simple linear regression.

Although the confidence-area test makes other types of significance tests unnecessary, one may still like to perform an analysis of variance (ANOVA) and apply the F-test (named in honour of R.A. Fisher). The following ANOVA procedure assumes that the regression is done of y (dependent variable) on x (independent variable).

#### **Symbols used**

y value of dependent variable

$\eta$  average value of y (mean)

SSD sum of squares of deviations

r correlation coefficient

R coefficient of explanation,  $R = 1 - \frac{\sum \delta^2}{\sum (y - \eta)^2}$ ,  
( $R > r^2$ )

df degrees of freedom

n number of (x,y) data sets

x independent variable

Var variance or “mean square of deviation”, it is the square value of the standard error  
( $\text{Var} = \text{SSD}/\text{df}$ )

$\delta$  residual after segmented linear regression with break-point, also called deviation from segmented linear regression

BP x-value of break-point

The term  $\sum (y - \eta)^2$  stands for “sum of squares of all reduced data”, briefly “reduced sum of squares”. It can be found from the SegReg output files, looking in the category of data with BP=0 (representing a linear regression of all data without break-point), using the value given for St.Dev.Y, then multiplying it with the total number of data minus 1, and finally calculating the square value of the result:

$$\Sigma(y-\eta)^2 = [(\text{St.Dev. } Y) \cdot (n-1)]^2$$

The value of  $r^2$  can be found directly in the same category of data. It is indicated by corr.coeff.sq. Note that  $r^2 = 1 - \Sigma \varepsilon^2 / \Sigma (y - \eta)^2$ , where  $\varepsilon$  is the residual after linear regression, also called deviation from linear regression.

In SegReg, the value of R is found in the group of results under “parameters for function type ...”. However, when the breakpoint is insignificant, only the regression without breakpoint is shown and the parameters are absent.

For variance analysis prepare the following table:

**Table 1.** ANOVA table for the segmented linear regression with breakpoint (SegReg Type 3 and 4 only)

Nr.	Description	SSD	df	Var
1.	explanation by simple linear regression without BP	$r^2 \Sigma(y-\eta)^2$	1	SSD <sub>1</sub>
2.	unexplained after linear regression (deviations from regression model)	$(1 - r^2) \Sigma(y-\eta)^2$	n-2	SSD <sub>2</sub> /(n-2)
3.	explanation by segmented linear regression with BP	$R \cdot \Sigma(y-\eta)^2$	-	SSD <sub>3</sub> (not used)
4.	additional explanation by BP analysis over simple linear regression	$(R - r^2) \Sigma(y-\eta)^2$	1	SSD <sub>4</sub>
5.	unexplained after BP analysis (deviations from regression model)	$(1 - R) \Sigma(y-\eta)^2$	n-3	SSD <sub>5</sub> /(n-3)
6.	total (initial, without regression)	$\Sigma(y-\eta)^2$	n-1	SSD <sub>6</sub> /(n-1) (not used)

If one wishes to test the significance of the simple linear regression one uses the F-statistic  $F(1, n-2) = \text{Var}_1 / \text{Var}_2$ . The F-statistic follows the F-distribution, named F in honour of R.A. Fisher. To be significant at probability level P, the F-statistic must be greater than the F-value found in F-tables at the probability P.

One may also wish to test if the additional explanation (4) could have arisen by chance. Hence the hypothesis is that BP analysis does not provide a significant extra contribution to the success of the simple linear regression. This is the “null-hypothesis”. If so, Var4 and Var5 both are independent estimates of the total variance Var6. The test-statistic under the null-hypothesis becomes  $F_0(1,n-3) = \text{Var4}/\text{Var5}$ . Under the null-hypothesis the  $F_0$ -value must be less than the F-value found in F-tables.

Some F-values at 5% probability of exceedance (or 95% non-exceedance,  $F_{0.95}$ ) are:

df4	$F_{0.95}$	Download F-test calculator from: <a href="http://www.waterlog.info/exe/f-testzip.exe">www.waterlog.info/exe/f-testzip.exe</a>
n-3 = 10	4.96	
n-3 = 20	4.35	
n-3 = 40	4.08	
n-3 = 120	3.92	
n-3 > 1000	3.84	

Now, if  $F_0 > F_{0.95}$  there is less than 5% chance that the BP analysis did not contribute significantly to the results. Taking a less than 5% risk, one may conclude that the null-hypothesis can be rejected and the conclusion is that the BP analysis has given a significant extra contribution to the regression compared to a simple linear regression (the BP is statistically “significant”).

On the other hand, when  $F_0 < F_{0.95}$ , the null-hypothesis is not rejected and the extra contribution is not significant, but one runs a risk of less than 5% that non-rejection of the null-hypothesis is unjustified.

## Notes

1. A major disadvantage of the F-test compared to the confidence-area test of BP, as done in SegReg, is that one obtains only a yes (BP is likely) or no (BP is doubtful) answer and one lacks an insight into the error-range to which the BP might be subjected even when it is significant.

$$2. \text{SSD}_3 = \text{SSD}_2 - \text{SSD}_1$$

$$\text{SSD}_5 = \text{SSD}_2 + \text{SSD}_4$$

## Reference

G.W. Snedecor and W.G. Cochran (1980), Statistical Methods, 7<sup>th</sup> edition, Chap. 19.3, p. 401-403. Iowa State University Press.

Chap. 19.3 deals with the “second degree polynomial”, which I have replaced by “segmented regression with break-point” as the principles are the same.

## Analysis of variance for data having several y-values for each x-value

### Additional symbols

a	number of classes or groups (minimum 4)
i	index indicating a group number ( $i = 1, 2, 3, \dots, a$ )
$\gamma_i$	mean value of y of group i
$\lambda_i$	value of y at $x_i$ according to segmented linear regression
$x_i$	x-value of group i
$j_i$	number of y-values in group i

**Table 2.** ANOVA table for data having several y-values for each x-value (Type 3 and 4 only)

Nr.	Description	SSD	df	Var
1.	explanation by simple linear regression without BP	$r^2 \Sigma(y-\eta)^2$	1	SSD <sub>1</sub> (not used)
2.	additional explanation by BP analysis over simple linear regression	$(R-r^2) \Sigma(y-\eta)^2$	1	SSD <sub>2</sub>
3.	explanation by segmented linear regression with BP <b>1 + 2</b>	$R \cdot \Sigma(y-\eta)^2$	2	SSD <sub>3</sub> /2 (not used)
4.	deviations of group means from segmented regression	$\Sigma(\gamma_i - \lambda_i)^2$	a-3	SSD <sub>4</sub> /(a-3)
5.	deviations between groups <b>3 + 4</b>	SSD <sub>3</sub> +SSD <sub>4</sub>	a-1	SSD <sub>5</sub> /(a-1)
6.	deviations within groups	$\Sigma_i \Sigma_{j_i} (y - \gamma_i)$	n-a	SSD <sub>6</sub> /(n-a)
7.	unexplained after regression with BP (deviations from regression model) <b>5 + 6</b>	$(1-R) \Sigma(y-\eta)^2$	n-3	SSD <sub>7</sub> /(n-3)
8.	total (initial, without regression)	$\Sigma(y-\eta)^2$	n-1	SSD <sub>8</sub> /(n-1) (not used)

The test-statistic  $F_0 = \text{Var}2/\text{Var}7$  derived from Table 2 has the same meaning as the statistic  $F_0 = \text{Var}4/\text{Var}5$  from Table 1.

The additional test-statistic is  $F_a = \text{Var}4/\text{Var}6$ , that can be compared to the value of  $F(a-3, n-a)$  in the F-tables. To conclude that the segmented regression is probably justified  $F_a$  needs to be the smaller one.

When the statistic  $F_a$  is significant, i.e.  $> F(a-3, n-a)$ , and  $F_0$  is significant (i.e. the contribution of BP is significant), the apparent non-linearity is probably not properly described by the present regression model.

## Reference

G.W. Snedecor and W.G. Cochran (1980), *Statistical Methods*, 7<sup>th</sup> edition, Chap. 19.4, p. 401-403. Iowa State University Press.

Chap. 19.4 deals with the "Data having several Ys for each X".

## Analysis of variance when two independent variables are present

When two independent variables (X and Z) are present, SegReg calculates the segmented linear regression with breakpoint (SLRB) of Y upon X and Y upon Z and it detects which of the two offers the highest degree of explanation. If this is true for X, it then performs an SLRB of the residuals YXr (after SLRB of Y upon X) on Z and the results are joined into one model. If, instead, Z offers the best fit after the first trial, the second SLRB is that of the residuals YZr (after SLRB of Y upon Z).

The sum of squares of deviations after a simple linear regression of residuals Yr (either YXr or YZr) on respectively Z or X can be found from

$$\Sigma(\varepsilon_1)^2 = [(\text{St.Dev. YXr}) \cdot (n-1)]^2$$

$$\Sigma(\varepsilon_2)^2 = [(\text{St.Dev. YZr}) \cdot (n-1)]^2$$

The values of the standard deviations of YXr or YZr are found in the SegReg output menu in the data category where BP=0 in the regression with residuals.

In the output file, SegReg also mentions in the same category the squares of the correlation coefficients ("corr.coeff.sq"), that we will call  $r_1^2$  and  $r_2^2$  for respectively a simple linear regression of YXr on Z and YZr on X.

The correlation coefficient calculated for the simple linear regression of Y upon X (as defined on page 1) or Y upon Z will be denoted by  $r_x$  and  $r_y$  respectively.

Similarly, the coefficient of explanation after SLRB of Y on X or Y on Z will be indicated by  $R_x$  and  $R_z$  respectively. Their values can be found in the output file of SegReg in the category "parameters for functions of type ..." just after the segmented linear regression with breakpoint of Y on X or Y on Z.

Further, the output file gives data on the coefficient of explanation ("expl.coeff") for the SLRB of either YXr on Z or YZr on X in the category "parameters for functions of type ....". These values we will call respectively  $R_1$  and  $R_2$ .

In the summary of the output file of Segreg, the overall coefficient explanation for the combined effect of the SLRB on X or Z and the residuals YZr and YXr respectively. This factor we will call  $R_t$ .

Now we can prepare an ANOVA table, depending on whether the first regression was done with the variable X or with the variable Z.

For the first case we get:

**Table 3.** ANOVA table for SLRB of Y upon X and the residuals YXr on Z

Nr.	Description	SSD	df	Var
1.	explanation by simple linear regression of Y on X without BP	$r_x^2 \Sigma(y-\eta)^2$	1	SSD <sub>1</sub>
2.	unexplained after linear regression of Y on X without BP (deviations from regression model)	$(1-r_x^2) \Sigma(y-\eta)^2$	n-2	SSD <sub>2</sub> /(n-2)
3.	explanation by segmented linear regression with BP of Y on X	$R_x \cdot \Sigma(y-\eta)^2$	-	SSD <sub>3</sub> (not used)
4.	additional explanation by BP analysis over simple linear regression of Y on X	$(R_x - r_x^2) \Sigma(y-\eta)^2$	1	SSD <sub>4</sub>
5.	unexplained after SLRB analysis of Y on X (deviations from regression model)	$(1-R_x) \Sigma(y-\eta)^2$	n-3	SSD <sub>5</sub> /(n-3)
6.	explanation by simple linear regression of YXr on Z without BP	$r_1^2 \Sigma(\varepsilon_1)^2$	1	SSD <sub>6</sub>
7.	unexplained after linear regression of YXr on Z without BP	$(1-r_1^2) \Sigma(\varepsilon_1)^2$	n-4	SSD <sub>7</sub> /(n-4)
8.	explanation by segmented linear regression with BP of YXr on Z	$R_2 \cdot \Sigma(\varepsilon_1)^2$	-	SSD <sub>8</sub> (not used)
9.	additional explanation by BP analysis over simple linear regression of YXr on Z	$(R_1 - r_1^2) \Sigma(\varepsilon_1)^2$	1	SSD <sub>9</sub>
10	unexplained after SLRB analysis of YXr on Z (deviations from regression model)	$(1-R_1) \Sigma(\varepsilon_1)^2$	n-5	SSD <sub>10</sub> /(n-5)

Now, the following test values of F can be prepared:

$F1(1,n-2) = \text{Var1}/\text{Var2}$ , to see if the simple linear regression of Y on X is significant

$F2(1,n-3) = \text{Var4}/\text{Var5}$ , to see if the SLRB of Y on X is significant

$F3(1,n-4) = \text{Var6}/\text{Var7}$ , to see if the simple linear regression of YXr on Z is significant

$F4(1,n-5) = \text{Var9}/\text{Var10}$ , to see if the SLRB of YXr on Z is significant.

For the second case we find:

**Table 4.** ANOVA table for SLRB of Y upon Z and the residuals YZr on X

Nr.	Description	SSD	df	Var
1.	explanation by simple linear regression of Y on Z without BP	$r_z^2 \Sigma(y-\eta)^2$	1	SSD <sub>1</sub>
2.	unexplained after linear regression of Y on Z without BP (deviations from regression model)	$(1-r_z^2) \Sigma(y-\eta)^2$	n-2	SSD <sub>2</sub> /(n-2)
3.	explanation by segmented linear regression with BP of Y on Z	$R_z \cdot \Sigma(y-\eta)^2$	-	SSD <sub>3</sub> (not used)
4.	additional explanation by BP analysis over simple linear regression of Y on Z	$(R_z - r_z^2) \Sigma(y-\eta)^2$	1	SSD <sub>4</sub>
5.	unexplained after SLRB analysis of Y on Z (deviations from regression model)	$(1-R_z) \Sigma(y-\eta)^2$	n-3	SSD <sub>5</sub> /(n-3)
6.	explanation by simple linear regression of YZr on X without BP	$r_2^2 \Sigma(\varepsilon_2)^2$	1	SSD <sub>6</sub>
7.	unexplained after linear regression of YZr on X without BP	$(1-r_2^2) \Sigma(\varepsilon_1)^2$	n-4	SSD <sub>7</sub> /(n-4)
8.	explanation by segmented linear regression with BP of YZr on X	$R_2 \cdot \Sigma(\varepsilon_2)^2$	-	SSD <sub>8</sub> (not used)
9.	additional explanation by BP analysis over simple linear regression of YZr on X	$(R_2 - r_2^2) \Sigma(\varepsilon_2)^2$	1	SSD <sub>9</sub>
10.	unexplained after SLRB analysis of YZr on X	$(1-R_2) \Sigma(\varepsilon_1)^2$	n-5	SSD <sub>10</sub> /(n-3)

Now, the following test values of F can be prepared:

$F1(1,n-2) = \text{Var1}/\text{Var2}$ , to see if the simple linear regression of Y on Z is significant

$F2(1,n-3) = \text{Var4}/\text{Var5}$ , to see if the SLRB of Y on Z is significant

$F3(1,n-4) = \text{Var6}/\text{Var7}$ , to see if the simple linear regression of YXz on X is significant

$F4(1,n-5) = \text{Var9}/\text{Var10}$ , to see if the SLRB of YXz on X is significant.

### Note

It is repeated that owing to the use of confidence intervals in SegReg, application of F-tests is not strictly necessary. For example, when SegReg finds that the introduction of a break-point gives no significant additional explanation, because the confidence interval of BP is too wide, it will not show a breakpoint. Also, when it is found that simple linear regression gives no significant explanation, because the confidence interval of the regression coefficient is too wide, it will not use the regression.

### Reference

G.W. Snedecor and W.G. Cochran (1980), *Statistical Methods*, 7<sup>th</sup> edition, Chap. 17.4, p. 401-403. Iowa State University Press.

Chap. 17.4 deals with “Extension of the analysis of variance in multiple linear regression to each individual explanatory variable”.