THE ENERGY BALANCE OF GROUNDWATER FLOW.

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ABSTRACT. An energy balance of groundwater flow is introduced. It is based on equating the change of hydraulic energy flux over a horizontal distance to the conversion rate of hydraulic energy into to friction of flow over that distance. The energy flux is calculated on the basis of a multiplication of the hydraulic potential with the flow velocity, and is integrated over the total flow depth. The conversion rate is determined in analogy to the heat loss equation of an electric current. The hydraulic energy balance is applied to the steady-state flow of water in a phreatic aquifer recharged by downward percolation stemming from rainfall or irrigation, and a quantitative example is given using a numerical solution. It is shown that the gradient of the water table is smaller than that calculated with the current methods, which do not take into account the energy associated with the incoming percolation water.

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1. ENERGY BALANCES

1.1 Energy fluxes

The hydraulic potential (P) can be defined as the energy per unit volume of water (ε/m^3, where ε represents energy units). The flow velocity (V) of groundwater can be defined as the discharge per unit cross-sectional area perpendicular to the direction of flow (m^3/day per m^2). The product P.V therefore represents an energy flux, i.e. an energy flow per unit cross-sectional area (ε/day per m^2).

Figure 1 shows a longitudinal section of two-dimensional groundwater flow (i.e. the flow pattern repeats itself in the planes parallel to the plane of the drawing) in a phreatic aquifer, i.e. an aquifer with a free water table. The water table is recharged by downward percolating water (R m/day) stemming from rainfall and/or irrigation. A coordinate system, with X (m) giving the horizontal and Z (m) the vertical distance from the origin, is also indicated. The horizontal component of the flow velocity in any point (X,Z) is indicated by Vx. The Z-levels of the impermeable layer and the water table in a vertical cross-section are shown as I and J respectively.

The total energy flow through a vertical cross-section (Ex, ε/day per m width in the direction perpendicular to the longitudinal section) is

$$Ex = \int_I^J \left[Vx(P-Pr) \right] dZ \qquad (1.1)$$

where Pr is a reference value of P, independent of X and Z, to be determined in accordance to the boundary conditions of the flow.

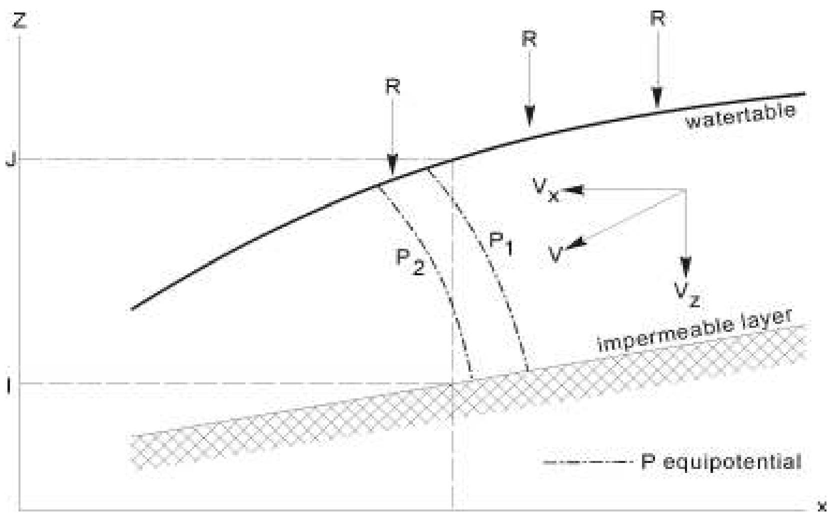

Figure 1. A vertical cross-section in a longitudinal section along two-dimensional groundwater flow in a phreatic aquifer recharged by percolation.

The change of the energy flow Ex per unit distance in a horizontal direction is

$$\frac{dEx}{dX} = \frac{d}{dX}\int_I^J [Vx(P-Pr)]\,dZ$$

Using Leibnitz's rule, and assuming that the impermeable layer is horizontal (I=0, dI/dX=0),the above equation can be written as

$$\frac{dEx}{dX} = \int_0^J \left[\frac{d}{dX}\{Vx(P-Pr)\}\right]dZ + Vj(Pj-Pr)\frac{dJ}{dX}$$

where Vj and Pj are the values of Vx and P at the water table.

Partial differentiation of the product Vx(P-Pr) in the previous Equation, and noting that dPr/dX=0, yields

$$\frac{dEx}{dX} = \int_0^J \left(Vx\frac{dP}{dX}\right)dZ + \int_0^J \left[(P-Pr)\frac{dVx}{dX}\right]dZ + Vj(Pj-Pr)\frac{dJ}{dX} \qquad (1.2)$$

1.2. The hydraulic head

The energy units ε, expressed in S.I. units, are $kg.m^2/day^2$, so that the units ε/m^3 of the potential P become kg/day^2 per m. The potential P can be converted into an hydraulic head as follows

$$H = P/\rho.g$$

where:

H is the hydraulic head (m)

ρ is the mass density of water (kg/m^3)

g is the gravitational acceleration (m/day^2)

With the conversion of potential P into head H, Equation 1.2 becomes

$$\frac{dEx/dX}{\rho.g} = \int_0^J \left(Vx\frac{dH}{dX}\right)dZ + \int_0^J \left[(H-Hr)\frac{dVx}{dX}\right]dZ + Vj(Hj-Hr)\frac{dJ}{dX} \qquad (1.3)$$

F4rom elementary hydraulics we know that the head H consists of three components: the elevation head (H_Z=Z), the pressure head (H_Y) and the velocity head (H_V). The velocity head of groundwater flow is negligibly small, so that H=Z+H_Y. At a phreatic surface, i.e. at the free water table, the pressure head corresponds to atmospheric pressure, which can be taken as zero reference pressure, so that H_Y=0. Hence, for Z=J, i.e. at the water table, we find $H_{Z=J}$=J.

Using the Dupuit assumptions that the velocity Vx, the head H, and the gradient dVx/dX are constant with height Z, so that Vj=Vx and H=$H_{Z=J}$=J, and writing Jr for Hr, Equation 1.3 can be simplified to

$$\frac{dEx/dX}{\rho.g} = (Vx.J)\frac{dJ}{dX} + J(J-Jr)\frac{dVx}{dX} + Vx(J-Jr)\frac{dJ}{dX} \qquad (1.4)$$

The assumption that the horizontal velocity Vx is constant with height is realistic when the resistance to vertical flow is small compared to that to horizontal flow.

1.3. The water balance

When the velocity Vx is constant with height Z, the two-dimensional discharge Q (m^3/day per m width of cross-section, or m^2/day) equals Q=Vx.J, and its differential coefficient, i.e. the change of discharge Q per unit change in distance X, becomes:

$$\frac{dQ}{dX} = \frac{d(Vx.J)}{dX} = Vx\frac{dJ}{X} + J\frac{dVx}{dX} = R$$

Hence, Equation 1.4 can be simplified to:

$$\frac{dEx/dX}{\rho.g} = (Vx.J)\frac{dJ}{dX} + R(J-Jr) \qquad (1.5)$$

1.4. Energy conversion by friction of flow

The electric current in a conduit is known to lose electrical energy by its conversion to heat. The conversion rate is proportional to the resistance of the conduit and the square value of the current (the law of Joule). The resistance is inversely proportional to the conductance. In analogy, the conversion of hydraulic energy to friction of flow is taken as

$$Fx = \int_0^J \left[\frac{(Vx)^2}{Kx}\right]dZ$$

where Kx is the horizontal hydraulic conductivity of the soil (m/day). Further, it is stated that the energy loss rate (as in Equation 1.5) is proportional to the negative value of the friction losses. Thus we obtain:

$$\frac{dE/dX}{\rho.g} = -Fx$$

Combining the previous two equations, and assuming again that the velocity Vx is constant with height Z, one obtains

$$\frac{dEx/dX}{\rho.g} = -J\frac{(Vx)^2}{Kx} \qquad (1.6)$$

1.5. The hydraulic energy balance for steady state flow

When there is no change in storage of water, and consequently there is no change in storage of hydraulic energy (i.e. energy storage associated with water storage), we have a steady state: the hydraulic energy losses are fully converted into frictional energy. It can then be found from Equation 1.5 and 1.6 that:

$$Vx \frac{dJ}{dX} + \frac{R(J-Jr)}{J} = - \frac{(Vx)^2}{Kx}$$

The minus sign in the above equation assures that the energy losses are positive when the gradient dJ/dX is negative, which occurs when the flow Vx is positive (i.e. in the positive x-direction or, in Figure 1, to the right), and vice versa. Division by Vx and rearrangement gives:

$$\frac{dJ}{dX} = - \frac{Vx}{Kx} - \frac{R(J-Jr)}{Vx.J} \qquad (1.7)$$

2. PHREATIC AQUIFERS WITH RECHARGE

2.1. The hydraulic energy balance equation

Figure 2 shows the two-dimensional flow of groundwater in a phreatic aquifer recharged by evenly distributed percolation from rainfall or irrigation (R>0, m/day). At the distance X=N (m) there is a water divide, here the water table is horizontal. The impermeable base is taken horizontal. The height of the water table above the impermeable base is taken equal to J (m). At the distance X≤N, the discharge of the aquifer equals Q=-R(N-X) (m^2/day). We find:

$$Vx = Q/J = -R(N-X)/J$$

With this, Equation 1.7 can be changed into

$$\frac{dJ}{dX} = \frac{R(N-X)}{Kx.J} - \frac{Jr-J}{N-X}$$

Setting F=J-Jo and Fr=Jr-J, where Jo is the value of J at X=0, and applying the condition that dF/dX=0 when X=N, we find Fr=Fn, where Fn is the value of F at X=N, and

$$\frac{dF}{dX} = \frac{R(N-X)}{Kx.J} - \frac{Fn-F}{N-X} \qquad (2.1)$$

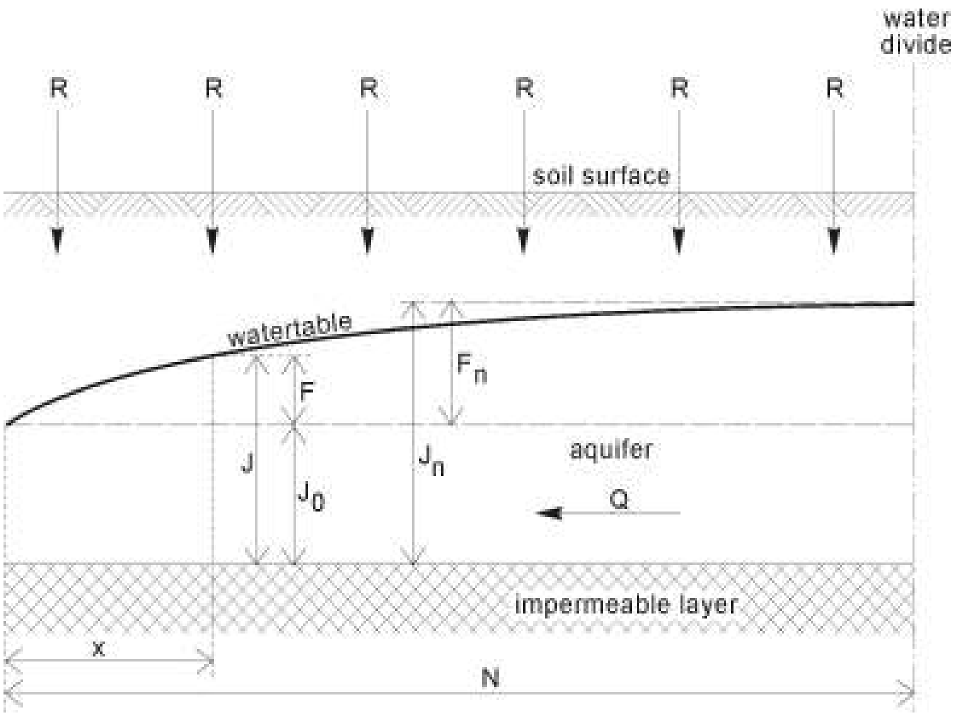

Figure 2. Flow conditions of groundwater in a phreatic aquifer recharged by percolation.

2.2. Integrations

Integrating Equation 2.1 from X=0, where F=0, to any value X, gives

$$F = \int_0^X \left[\frac{R(N-X)}{Kx.J}\right] dX - \int_0^X \left[\frac{Fn-F}{N-X}\right] dX \qquad (2.2)$$

The integration of the last term requires advance knowledge of Fn. To overcome this problem, a numerical solution and a trial and error procedure is given in Section 2.4.

2.3. The current method of analysis

When, according to the current method of analysis, the Darcy equation is used with the water balance and the Dupuit assumptions to describe the groundwater flow under the same conditions, one finds instead of Equation 2.2 (e.g. Wesseling 1973):

$$F^* = \int_0^X \left[\frac{R(N-X)}{Kx.J}\right] dX \qquad (2.3)$$

Here, the symbol F* is used instead of F to indicate the current method of analysis. In the following, a numerical solution of Equation 2.4 is given, but the equation can also be solved directly as

$$F_n^* (J_0 + ½F_n^*) = ½R.N^2/Kx \qquad (2.4)$$

2.4. Numerical integrations

For the numerical integrations, the horizontal distance N is divided into a number (T) of equal small elements with length U, so that U=N/T. The elements are numbered S = 1, 2, 3, ..., T. The heights F and J of the water table in the point defined by the largest value of distance X in element S are denoted as F_S and J_S. The change of height F over the S-th element in the zone of radial flow is denoted as G_S and found from

$$G_S = F_S - F_{S-1}$$

The average value of height F over the S-th element is

$$\underline{F}_S = F_{S-1} + ½G_{S-1}$$

and the average of the cross-sectional height J of flow is

$$\underline{J}_S = J_{S-1} + ½G_{S-1}$$

The average value of the horizontal distance X of the S-th element from the center of the drain is found as

$$X_S = U(S - 0.5)$$

Equation 2.1 can now be approximated by:

$$G_S = U(A_S+B_S) \qquad (2.5)$$

where

$$A_S = R.(N-X_S)/K.\underline{J}_S$$

$$B_S = (\underline{F}_S-F_T)/(N-X_S)$$

where F_T is the value of F_S when S=T. Now, the height of the water table at any distance X can be found, conform to Equation 2.2, from:

$$F_S = \sum_{1}^{S} G_S \qquad (2.6)$$

Since F_S depends on B_S and B_S on F_T, which is not known in advance, Equation 2.6 must be solved by trial and error.

In similarity to the above procedure, the value G_S* (where the symbol * is used to indicate the numerical solution of Equation 2.3 instead of 2.2, i.e. not using the energy balance but the current method of analysis) is found as

$$G_S^* = R.U(N-X_S)/K.\underline{J}_S^* \qquad (2.7)$$

where $\underline{J}_S^*=J_{S-1}^*+½G_{S-1}^*$. Thus the height of the water table, in conformity to Equation 2.4, is:

$$F_S^* = \sum_{1}^{S} G_S^* \qquad (2.8)$$

2.5. Example of a numerical solution

To illustrate the numerical solutions we use the following data:

N	=	100.0	m	J_0	=	10.0	m
Kx	=	1.0	m/day	R	=	0.01	m/day
U	=	0.5	m				

For the example, the calculations with Equations 2.5, 2.6, 2.7 and 2.8 were made on a computer. The results are presented in the Tables 1 and 2 and in Figure 3.

Table 1 gives the values of height F_S (m) and gradients G_S/p, A_S, B_S at some selected values of distance X, using Equations 2.5 and 2.6 (i.e. using the energy balance) with steps of U=0.5 m, so that in total 200 steps are taken with a large number of iterations per step. Smaller values of step U do not yield significantly different results.

Table 2 gives the values of height F_S* and gradient G_S*/p, at the same selected values of distance X of Table 1 and 2, using Equations 2.7 and 2.8 (i.e. ignoring the energy balance). It is seen from Table 2.2 that the Fn* value (i.e the value of F* at X=N=100 m) equals 4.142 m. This is in agreement with the value Fn*=4.142 m that can be calculated directly from Equation 2.4.

Comparison of the tables learns that the Fn value (i.e. the value of F at X=N=100 m) of Table 1 (Fn=2.972) is considerably smaller than the Fn* value (4.142 m) of Table 2 (i.e. without energy balance). This is also shown in Figure 3.

(Postscript. The computer program used was later refined and it uses steps of U=0.01 m standard. The program is available under the name of EnDrain, see www.waterlog.info/endrain.htm)

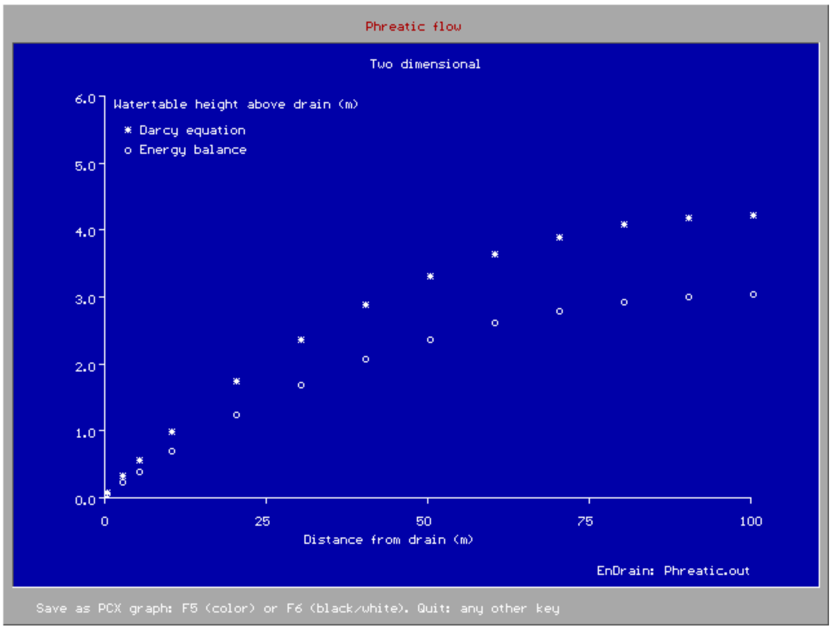

Figure 3. The shape of the water table calculated with the energy balance equation and the Darcy equation for the conditions given in the example.

Table 1. Results of the calculations of the height of the Water table using a numerical and iterative solution of the hydraulic energy balance with steps U=0.5 m for the conditions described the example, using Equations 2.5 and 2.6.

Distance X (m)	Height of the water-table F (m)	Gradient of F G/U (m/m)	Gradient needed for the flow A (m/m)	Adjustment of A due to the energy of recharge B (m/m)
1	0.070	0.069	0.099	-0.029
2	0.138	0.068	0.097	-0.029
5	0.336	0.064	0.092	-0.028
10	0.643	0.059	0.085	-0.026
20	1.181	0.049	0.072	-0.022
30	1.630	0.041	0.060	-0.019
40	2.004	0.034	0.050	-0.016
50	2.310	0.027	0.041	-0.013
60	2.553	0.021	0.032	-0.011
70	2.739	0.016	0.024	-0.008
80	2.869	0.010	0.016	-0.005
90	2.947	0.005	0.008	-0.003
95	2.966	0.003	0.004	-0.001
98	2.971	0.001	0.002	-0.001
99	2.972	0.001	0.001	-0.000
100 (N)	2.972	0.000	0.000	-0.000

Table 2. Results of the calculations of the level of the water table using a numerical solution of Equation 2.7 and 2.8 (i.e. without energy balance), with steps U=0.5 m, for the conditions described in the example.

Distance X (m)	Height of the water table F* (m)	Gradient of F* G*/U (m/m)
1	0.099	0.099
2	0.196	0.097
5	0.476	0.092
10	0.909	0.083
20	1.662	0.069
30	2.288	0.058
40	2.806	0.047
50	3.229	0.038
60	3.564	0.030
70	3.820	0.022
80	4.000	0.015
90	4.107	0.007
95	4.133	0.004
98	4.141	0.002
99	4.142	0.001
100 (N)	4.142	0.000

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